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$K \to \pi\ell\nu$ Semileptonic Form Factors from Two-Flavor Lattice QCD

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We present new lattice results of the $K \to \pi\ell\nu$ semileptonic form factors obtained from simulations with two flavors of dynamical twisted-mass fermions, using pion masses as light as 260 MeV. Our main result is $f_+(0) = 0.9560(84)$, which, combined with the latest experimental data for $K_{\ell 3}$ decays, leads to $|V_{us}| = 0.2267(5)_{\rm exp.}(20)_{f_+(0)}$. Using the PDG(2008) determinations of $|V_{ud}|$ and $|V_{ub}|$ our result implies for the unitarity relation $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1.0004(15)$. For the $O(p^6)$ term of the chiral expansion of $f_+(0)$ we get $\Delta f \equiv f_+(0) - 1 - f_2 = -0.0214(84)$.

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The Cabibbo's angle, or equivalently the CKM matrix element $|V_{us}|$ [1], is one of the fundamental parameters of the Standard Model. The most precise determination of $|V_{us}|$ comes from $K \to \pi\ell\nu$ ($K_{\ell 3}$) decay. The PDG(2008) quotes $|V_{us}| = 0.2255(19)$ [2]. It is based on the new, very accurate experimental determination of the product $|V_{us}|f_+(0) = 0.21668(45)$ [2, 3] and on the old estimate of the vector form factor at zero-momentum transfer $f_+(0) = 0.961(8)$ given in Ref. [4].

The determination of $f_+(0)$ using lattice QCD started only recently with the quenched calculation of Ref. [5], where it was shown how $f_+(0)$ can be determined at the physical point with a $\simeq 1\%$ accuracy. The findings of Ref. [5] triggered various unquenched calculations of $f_+(0)$, namely those of Refs. [6, 7, 8] with $N_f = 2$ and pion masses above $\simeq$ 500 MeV and the recent one of Ref. [9] with $N_f = 2 + 1$ and pion masses starting from 330 MeV.

In this Letter we present a new lattice result for $f_+(0)$ obtained from simulations with two flavors of dynamical twisted-mass quarks, using pion masses from 260 MeV up to 580 MeV. Our determination of $f_+(0)$ includes the estimates of all sources of systematic errors: discretization, finite size effects (FSE's), q^2-dependence, chiral extrapolation and the effects of quenching the strange quark.

The chiral extrapolation and the related uncertainty are investigated using both SU(3) and, for the first time, SU(2) Chiral Perturbation Theory (ChPT). Within the former one can perform a systematic expansion of $f_+(0)$ of the type $f_+(0) = 1 + f_2 + f_4 + ...$, where $f_n = \mathcal{O}[M^n_{K,\pi}/(4\pi f_\pi)^n]$ and the first term is equal to unity due to the current conservation in the SU(3) limit. Because of the Ademollo-Gatto (AG) theorem [10], the first correction f_2 does not receive contributions from the local operators of the effective theory and can be computed unambiguously in terms of the kaon and pion masses (M_K and M_π) and the pion decay constant f_π. It takes the value $f_2 = -0.0226$ at the physical point [4]. The task is thus reduced to the problem of finding a prediction for

$$\Delta f \equiv f_4 + f_6 + ... = f_+(0) - (1 + f_2) \ . \tag{1}$$

Recently SU(2) ChPT at the next-to-leading order (NLO) has been applied to study the quark-mass dependence of $f_+(0)$ [11]. In SU(2) ChPT the strange quark field does not satisfy chiral symmetry and the dependence on the strange quark mass, m_s, is absorbed into the low-energy constants (LEC's) of the effective theory. The convergence of SU(2) ChPT is expected to be good when the u/d quark mass is significantly smaller than m_s. In the case of $f_+(0)$ one gets the NLO result [11]

$$f_+(0) = F_+ - \frac{3}{4}\frac{M_\pi^2}{(4\pi f_\pi)^2}\log(\frac{M_\pi^2}{\mu^2}) + c_+ M_\pi^2 + \mathcal{O}(M_\pi^4) \tag{2}$$

where F_+ and c_+ are LEC's functions of m_s and c_+ depends also on the renormalization scale μ in such a way that the whole NLO result (2) is independent on μ.

For the extrapolation of our lattice data to the physical point we apply both SU(2) and SU(3) ChPT obtaining consistent results, which help constraining the uncertainty of the chiral extrapolation.

We perform simulations with $N_f = 2$ flavors of dynamical twisted-mass quarks [12] generated with the tree-level Symanzik improved gauge action at a lattice spacing $a = 0.0883(6)$ fm [13, 14] ($\beta = 3.9$), for six values of the (bare) sea quark mass, namely $am_{sea} = 0.0030, 0.0040, 0.0064, 0.0085, 0.0100, 0.0150$ (see Ref. [15]). The valence light-quark mass is always kept equal to the sea quark mass (unitary pions) and the simulated pion masses goes from $\simeq$ 260 to $\simeq$ 575 MeV. For each pion mass we use three values of the (bare) strange quark mass, namely $am_s = 0.015, 0.022, 0.027$, to allow for a smooth, local interpolation of our results to the physical strange quark mass ($am_s^{phys} \simeq 0.021$).

At the two lowest pion masses the lattice volume is $L^3 \cdot T = 32^3 \cdot 64$ in lattice units, while at the higher ones it is $24^3 \cdot 48$ in order to guarantee that $M_\pi L \gtrsim 3.7$.

We perform two additional simulations: the first one at $M_\pi \simeq 300$ MeV using the smaller volume and the second at $M_\pi \simeq 470$ MeV using a finer lattice spacing $a \simeq 0.07$ fm ($\beta = 4.05$) in order to check FSE's and discretization errors, respectively.

The 2- and 3-point correlation functions relevant in this work are calculated using all-to-all quark propagators evaluated with the "one-end-trick" stochastic procedure. All the necessary formulae can be easily inferred from Ref. [13], where the degenerate case of the pion form factor is illustrated in details. At each value of the pion mass the statistical errors are evaluated with the jackknife procedure, while a bootstrap sampling is applied in order to combine the jackknives for different pion masses.

The matrix element of the weak vector current V_μ can be written as

$$\langle \pi(p')|V_\mu|K(p)\rangle \; = \; P_\mu \; f_+(q^2) + q_\mu \; f_-(q^2) \; , \qquad (3)$$

where $P_\mu = p_\mu + p'_\mu$ and $q_\mu = p_\mu - p'_\mu$, and the scalar form factor $f_0(q^2)$ is defined as

$$f_0(q^2) \; = \; f_+(q^2) + \frac{q^2}{M_K^2 - M_\pi^2} f_-(q^2) \; . \qquad (4)$$

Following Ref. [5] the scalar form factor at $q^2 = q^2_{max} \equiv (M_K - M_\pi)^2$ can be calculated on the lattice with very high statistical precision using a suitable double ratio of 3-point correlation functions. In the present simulations we get a precision better than $\simeq 0.2\%$ (see Table 1).

At each pion and kaon masses we determine both the vector $f_+(q^2)$ and the scalar $f_0(q^2)$ form factors for several values of $q^2 < q^2_{max}$ in order to interpolate at $q^2 = 0$. We take advantage of the twisted boundary conditions (see Ref. [13] for details) to achieve values of q^2 quite close to $q^2 = 0$. The momentum dependencies of both form factors are nicely fitted either by a pole behavior

$$f_{+,0}(q^2) = f_+(0)/(1 - s_{+,0} \; q^2) \qquad (5)$$

or by a quadratic dependence on q^2

$$f_{+,0}(q^2) = f_+(0) \cdot (1 + \bar{s}_{+,0} \; q^2 + \bar{c}_{+,0} \; q^4) \; , \qquad (6)$$

where the condition $f_0(0) = f_+(0)$ is understood. The quality of the two fits is illustrated in Fig. 1.

The values obtained for $f_+(0)$ depend on both the pion and kaon masses. The dependence on the latter is shown in Fig. 2 at $M_\pi \simeq 435$ MeV and it appears to be quite smooth, so that an interpolation at the physical strange quark mass can be easily performed using quadratic splines. This is obtained by fixing the combination $(2M_K^2 - M_\pi^2)$ at its physical value, which at each pion mass defines a *reference* kaon mass M_K^{ref}:

$$2[M_K^{ref}]^2 - M_\pi^2 = 2[M_K^{phys}]^2 - [M_\pi^{phys}]^2 \qquad (7)$$

with $M_\pi^{phys} = 135.0$ MeV and $M_K^{phys} = 494.4$ MeV.

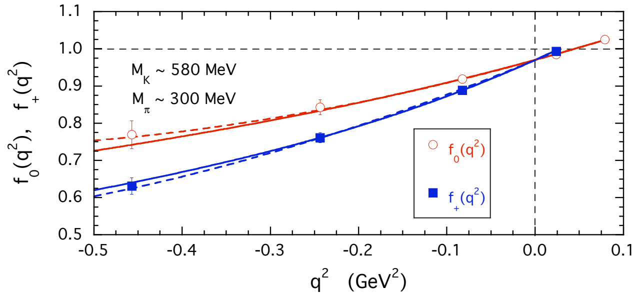

Fig. 1: *Scalar $f_0(q^2)$ and vector $f_+(q^2)$ form factors obtained at $M_\pi \simeq 300$ MeV and $M_K \simeq 580$ MeV versus q^2 in physical units. The solid and dashed lines are the results of the fits based on Eqs. (5) and (6), respectively.*

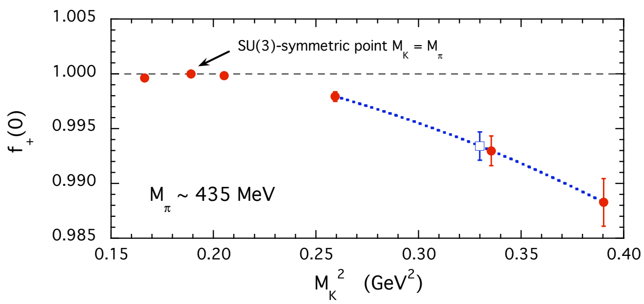

Fig. 2: *Results for $f_+(0)$ versus M_K^2 at $M_\pi \simeq 435$ MeV. The square corresponds to the value of $f_+(0)$ obtained by local interpolation via quadratic splines (dotted line) at the reference kaon mass $M_K^{ref} \simeq 575$ MeV from Eq. (7).*

Note that at the SU(3)-symmetric point $M_K = M_\pi$ the absolute normalization $f_+(0) = 1$ is imposed automatically by the double ratio method of Ref. [5].

The results for $f_+(0)$, obtained using the pole dominance (5) or the quadratic fit (6), and interpolated at the reference kaon mass (7), are given in Table 1 for each pion mass. It can be seen that the values of $f_+(0)$ corresponding to different q^2-dependencies of the form factors differ by less than half of the statistical errors. In what follows we will show in the figures only the results obtained using the pole fit (5).

M_π (MeV)	M_K^{ref} (MeV)	$f_0(q^2_{max})$	$f_+(0)$ (pole)	$f_+(0)$ (quadratic)
260	520	1.03097 (224)	0.97519 (499)	0.97374 (505)
300	530	1.01923 (121)	0.98052 (440)	0.97950 (390)
375	555	1.00961 (123)	0.98916 (264)	0.98813 (248)
435	575	1.00416 (43)	0.99343 (130)	0.99273 (131)
470	590	1.00272 (34)	0.99421 (79)	0.99413 (85)
575	635	1.00016 (6)	0.99823 (15)	0.99827 (19)

Table 1: *Results for $f_0(q^2_{max})$ and $f_+(0)$, obtained with the pole (5) or quadratic (6) fits, interpolated at the reference kaon mass (7) for each simulated pion mass.*

The SU(3) chiral analysis of $f_+(0)$ starts by considering

the NLO term f_2, using the exact expression f_2^{PQ} evaluated for our partially quenched (PQ) setup in Ref. [16],

$$f_2^{PQ} = -\frac{2M_K^2+M_\pi^2}{32\pi^2 f_\pi^2} - \frac{3M_K^2 M_\pi^2 \log(M_\pi^2/M_K^2)}{64\pi^2 f_\pi^2 (M_K^2-M_\pi^2)} + \frac{M_K^2(4M_K^2-M_\pi^2)\log(2-M_\pi^2/M_K^2)}{64\pi^2 f_\pi^2(M_K^2-M_\pi^2)} , \quad (8)$$

and by constructing the quantity Δf from Eq. (1). We then carry out the extrapolation to the physical point using a simple phenomenological ansatz in terms of M_π^2:

$$\Delta f = \Delta_0 + \Delta_1 M_\pi^2 + \Delta_2 M_\pi^4 + \Delta_3 M_\pi^2 \log(M_\pi^2) , \quad (9)$$

where $\Delta_{0,1,2,3}$ are fitting parameters.

The results obtained for $f_+(0)$ using two fits for Δf, one with $\Delta_3 = 0$ and the other with $\Delta_2 = 0$, are shown in Fig. 3(a). It can be seen that: i) the (absolute) size of Δf, whose chiral expansion starts from the NNLO term f_4, is even larger than the one of the leading NLO term f_2^{PQ} at all pion masses, and ii) the impact of the logarithmic term at NNLO is quite small.

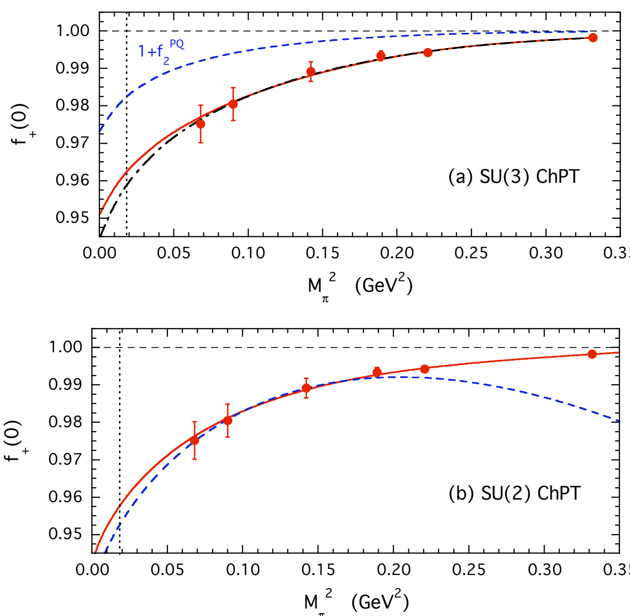

Fig. 3: *Results for $f_+(0)$ versus M_π^2 at $M_K = M_K^{ref}$ analyzed using SU(3) (a) and SU(2) (b) ChPT. In (a) the SU(3) LO + NLO term, $1+f_2^{PQ}$, is shown by the dashed line. The solid and dot-dashed lines are the results of the fit (9) for Δf with $\Delta_3 = 0$ and $\Delta_2 = 0$, respectively. In (b) the dashed line is the SU(2) LO + NLO fit (2) applied to our data with $M_\pi \lesssim$ 0.4 GeV, while the solid line corresponds to the result of fitting all lattice points adding to Eq. (2) a NNLO term proportional to M_π^4. The vertical line corresponds to $M_\pi^{phys} = 135.0$ MeV.*

A relevant check on our fits (9) is that they turn out to be consistent with zero (within the statistical errors) at the point $M_\pi = M_K^{ref}$, as required by the AG theorem.

At the physical point we get

$$f_+(0)|_{SU(3)}^{PQ} = 0.9599(61)(32) , \quad (10)$$

where the first error is statistical and the second one is systematic coming from the uncertainties of the mass extrapolation and the q^2-dependence of the form factors.

We now discuss the analysis based on SU(2) ChPT. First we note that Eq. (2) holds for full QCD [11] as well as for the PQ theory with $N_f = 2$. In the latter case it can be verified by expanding f_2^{PQ} [see Eq. (8)] in powers of M_π^2/M_K^2. Thus we consider a SU(2) fit of the form (2) treating F_+ and c_+ as fitting parameters, and we apply it to our data with $M_\pi \lesssim 0.4$ GeV. Alternatively we add to Eq. (2) a NNLO correction proportional to M_π^4 and apply the new fit to all lattice points. The results are shown in Fig. 3(b). It can be seen that the impact of the SU(2) NNLO correction is quite small up to $M_\pi \approx 0.5$ GeV at variance with the corresponding SU(3) result shown in Fig. 3(a). This finding signals a better convergence of SU(2) ChPT with respect to SU(3) for $f_+(0)$.

At the physical point we get

$$f_+(0)|_{SU(2)}^{PQ} = 0.9563(53)(13) . \quad (11)$$

The application of SU(2) and SU(3) ChPT yields results for $f_+(0)$, Eqs. (10) and (11), which are consistent within the uncertainties. By averaging the two results and adding the systematic errors in quadrature we get

$$f_+^{PQ}(0) = 0.9581 \pm 0.0057_{\text{stat.}} \pm 0.0035_{\text{syst.}} . \quad (12)$$

We now present our estimates of the remaining sources of systematic effects.

Finite Size. We have performed a simulation at $M_\pi = 300$ MeV using the volume $24^3 \cdot 48$, which corresponds to $M_\pi L \simeq 3.2$. We get $f_+(0) = 0.98633(362)$ using the pole-dominance fit (5) and $f_+(0) = 0.98597(337)$ using the quadratic fit (6). We combine these values with the results shown in the second row of Table 1, corresponding to the volume $32^3 \cdot 64$ with $M_\pi L \simeq 4.2$. Assuming a volume dependence of the form $A + Be^{-M_\pi L}/L^{3/2}$ we obtain a residual FSE equal to 0.0018, which we add (in quadrature) to the systematic error of Eq. (12).

Discretization. We have performed a simulation at $M_\pi \simeq 470$ MeV using a finer lattice spacing ($a \simeq 0.07$ fm). We observe a systematic increase of the scalar form factor $f_0(q^2)$ at all values of q^2 and for all kaon masses. In particular we get $f_+(0) = 0.99555(80)$ and 0.99518(95) using the pole-dominance (5) and the quadratic (6) fits, respectively. We combine these values with the results shown in the fifth row of Table 1. Assuming a linear fit in a^2 (which is consistent with the automatic $\mathcal{O}(a)$ improvement at maximal twist [17]), we find a discretization effect equal to 0.0037, which we add both to the central value and (in quadrature) to the systematic error of Eq. (12). Clearly a more detailed study of the scaling property of $f_+(0)$ would be beneficial in order to estimate better and to reduce further the discretization error.

Quenching of the strange quark. The effect of our PQ setup can be estimated within SU(3) ChPT, because, thanks to the AG theorem, the effect of quenching the strange quark is exactly known at NLO: at the physical point $f_2 - f_2^{PQ} = -0.0058$ ($\simeq 26\%$ of f_2). This correction is added to the central value of Eq. (12). As for the $\mathcal{O}(p^6)$ term Δf, we have found evidence that the chiral logs, which are the most sensitive to quenching effects, are small compared to the contribution of the local terms (see Fig. 3(a)). We estimate that the quenching effect on Δf is at most 50% of the same effect on f_2. Thus we add (in quadrature) the value 0.0028 ($\simeq 13\%$ of Δf) to the systematic error of Eq. (12). Note that this value is of the same size of the difference between our estimate of Δf at $N_f = 2$ and the quenched one of Ref. [5].

Our final result is

$$\begin{aligned} f_+(0) &= 0.9560 \pm 0.0057_{\text{stat.}} \pm 0.0062_{\text{syst.}} \\ &= 0.9560 \pm 0.0084 \ , \end{aligned} \tag{13}$$

which corresponds to $\Delta f = -0.0214(84)$. Our determination agrees very well with the Leutwyler-Roos result [4] and with previous lattice calculations at $N_f = 0$ [5], $N_f = 2$ [6, 7, 8] and $N_f = 2 + 1$ [9].

Using the latest experimental determination of the product $|V_{us}| f_+(0) = 0.21668(45)$ [2, 3] we get from (13)

$$|V_{us}| = 0.2267 \pm 0.0005_{\text{exp.}} \pm 0.0020_{f_+(0)} \ . \tag{14}$$

Combining this value with $|V_{ud}| = 0.97418(27)$ and $|V_{ub}| = 0.00393(36)$ from PDG2008 [2] the CKM unitarity relation becomes

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1.0004 \pm 0.0015. \tag{15}$$

In conclusion we present our results for the slopes of the scalar (s_0) and vector (s_+) form factors. Their light-quark mass dependence is illustrated in Fig. 4 and it appears to be quite mild. We have tried simple fitting functions of the form

$$s_j = a_j + b_j M_\pi^2 + c_j M_\pi^4 + d_j M_\pi^2 \log(M_\pi^2) \ , \tag{16}$$

where a_j, b_j, c_j and d_j are fitting parameters and $j = +, 0$. The results of two fits, one with $d_{+,0} = 0$ and the other with $c_{+,0} = 0$, are shown in Fig. 4.

In terms of the dimensionless quantities $\lambda_{+,0} \equiv M_\pi^2 \ s_{+,0}$ the extrapolation to the physical point and the evaluation of the systematic uncertainties yield

$$\begin{aligned} \lambda_0 &= (12.8 \pm 2.2_{\text{stat.}} \pm 4.5_{\text{syst.}}) \cdot 10^{-3} \ , \\ \lambda_+ &= (23.7 \pm 2.3_{\text{stat.}} \pm 2.1_{\text{syst.}}) \cdot 10^{-3} \ , \end{aligned} \tag{17}$$

where the large systematic error on λ_0 is dominated by discretization effects. Our results for both λ_0 and λ_+ agree very well with the latest experimental averages $\lambda_0^{exp.} = (13.4 \pm 1.2) \cdot 10^{-3}$ and $\lambda_+^{exp.} = (24.9 \pm 1.1) \cdot 10^{-3}$, obtained in Ref. [3] using data from KLOE, KTeV, ISTRA+ and NA48 experiments.

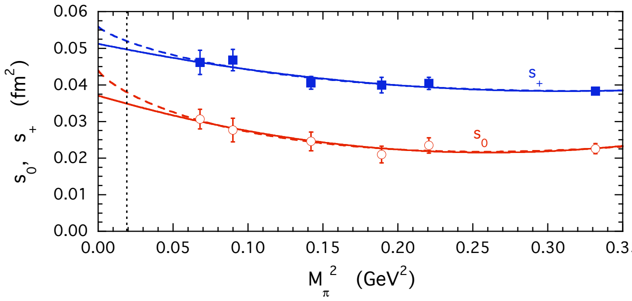

Fig. 4: *Results for the slopes s_0 (dots) and s_+ (squares) versus M_π^2 at $M_K = M_K^{ref}$. The solid (dashed) line corresponds to the fit (16) with $d_{+,0} = 0$ ($c_{+,0} = 0$).*

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